

# Interplay between curvature and Planck-scale effects in astrophysics and cosmology

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## ABSTRACT

Several recent studies have considered the implications for astrophysics and cosmology of some possible nonclassical properties of spacetime at the Planck scale. The new effects, such as a Planck-scale-modified energy-momentum (dispersion) relation, are often inferred from the analysis of some quantum versions of Minkowski spacetime, and therefore the relevant estimates depend heavily on the assumption that there could not be significant interplay between Planck-scale and curvature effects. We here scrutinize this assumption, using as guidance a quantum version of de Sitter spacetime with known Inönü-Wigner contraction to a quantum Minkowski spacetime. And we show that, contrary to common (but unsupported) beliefs, the interplay between Planck-scale and curvature effects can be significant. Within our illustrative example, in the Minkowski limit the quantum-geometry deformation parameter is indeed given by the Planck scale, while in the de Sitter picture the parameter of quantization of geometry depends both on the Planck scale and the curvature scalar. For the much-studied case of Planck-scale effects that intervene in the observation of gamma-ray bursts we can estimate the implications of “quantum spacetime curvature” within robust simplifying assumptions. For cosmology at the present stage of the development of the relevant mathematics one cannot go beyond semiheuristic reasoning, and we here propose a candidate approximate description of a quantum FRW geometry, obtained by patching together pieces (with different spacetime curvature) of our quantum de Sitter. This semiheuristic picture, in spite of its limitations, provides rather robust evidence that in the early Universe the interplay between Planck-scale and curvature effects could have been particularly significant.

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## I. INTRODUCTION

For many decades [1, 2] progress in the study of the quantum-gravity problem was obstructed by the extreme mathematical complexity of the most promising theories of quantum gravity, resulting in a debate that was confined at the level of comparison of mathematical and conceptual features. At least for one aspect of the quantum-gravity problem, the one that concerns the possibility that spacetime itself might have to be quantized, the nature of the debate started to change in the second half of 1990s when it was established that some scenarios for the quantization of spacetime have implications for spacetime symmetries, which have then been studied focusing mainly on the aspects of modification of the classical-spacetime “dispersion” relation between energy and momentum of a microscopic particle. These developments have also motivated a rather large effort on the side of phenomenology (see, *e.g.*, Refs. [3–13]) looking for ways to gain experimental insight on this hypothesis both in laboratory experiments and, even more frequently, using astrophysics observatories.

Cosmology has so far played only a relatively marginal role in this phenomenology research effort, but it appears likely that this might change in the not-so-distant future. This expectation originates from the fact that many cosmological observations reflect the properties of the Universe at very early times, when the typical energies of particles were significantly closer to the Planck scale than the energies presently reached in our most advanced particle accelerators. Moreover, the particles studied in cosmology have typically travelled ultra-long (“cosmological”) distances, and therefore even when they are particles of relatively low energies they could be affected by a large accumulation of the effects of the “space-time quantization”, which is one of the most common expectations emerging from quantum-gravity research.

We are here mainly concerned with a key assumption that is commonly made in the few studies of quantum-spacetime effects for cosmology that have been produced so far (see, *e.g.*, Refs. [14–20]). This is basically the assumption that the quantum-spacetime effects could be safely estimated in quantum versions of Minkowski spacetime, and then inserted “by hand” as new features for the analysis in cosmology, which of course is not formulated in Minkowski spacetime. For example, for what concerns the energy-momentum relation, one essentially assumes that, if in the Minkowski limit the energy-momentum relation is of the type<sup>1</sup>  $m^2 = P^\mu \eta_{\mu\nu} P^\nu + F_{flat}(L_p, P^\alpha)$ , in cases with metric  $g_{\mu\nu}$  ( $\neq \eta_{\mu\nu}$ ) one could still write

$$m^2 = P^\mu g_{\mu\nu} P^\nu + F_{flat}(L_p, P^\alpha), \quad (1)$$

with the same deformation function  $F_{flat}(L_p, P^\alpha)$ .

We here investigate this issue of the interplay between curvature and Planck-scale effects within the framework that was introduced for these purposes in Ref. [21] (also see Ref. [22]), which advocated the study of a specific example of quantum de Sitter (dS) spacetime, with known Inönü-Wigner contraction to a much-studied quantum Minkowski spacetime. We find that the interplay between curvature and Planck-scale effects is very significant, and in particular our analysis produces candidates for relations of the type

$$m^2 = P^\mu g_{\mu\nu}^\Lambda P^\nu + F(\Lambda, L_p, P^\alpha), \quad (2)$$

where  $g_{\mu\nu}^\Lambda$  is the dS metric for cosmological constant  $\Lambda$ . The significance of the interplay between curvature and Planck-scale effects admits in our framework a particularly straightforward description: our quantum version of dS spacetime is dual to a Hopf algebra whose characteristic parameter is dimensionless. So the only opportunities for the Planck scale to appear in the description of the structure of our quantum spacetime necessarily involves expressing this dimensionless parameter in terms of the Planck scale and of the only other dimensionful scale present in the framework, which is indeed the curvature scalar.

While our main technical findings concern a candidate for a quantum dS spacetime, we argue that at least at a semi-heuristic/semi-quantitative level they are valuable also for some (yet to

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<sup>1</sup> In this work we set  $c = \hbar = 1$ .

be formalized) quantum versions of FRW geometries. We structure this aspect of our thesis by introducing an approximate description of a quantum FRW geometry, obtained by patching together pieces (with different curvature) of our quantum dS. The quantum-deformation parameter characteristic of our setup must be specified as a function of the Planck scale and of the “effective dS-patch curvature”, and we find that different formulations of this relation (all with the same Minkowski limit!) lead to very different descriptions of the path of massless particles. We therefore provide an explicit example of the significance of the interplay between curvature and Planck-scale effects in cosmology.

In preparation for the main parts of the analysis, in the next section we briefly review some well-known aspects of the classical dS spacetime, mainly establishing notation to be used in the following. Then in Section III we introduce our quantum version of dS spacetime, a “q-dS spacetime”, and its contraction to the  $\kappa$ -Minkowski noncommutative spacetime.  $\kappa$ -Minkowski is a relevant example since it has inspired some of the studies considering Planck-scale effects in astrophysics and cosmology. Our q-dS spacetime is a natural generalization of  $\kappa$ -Minkowski to the case of a constant-curvature maximally-symmetric spacetime. We work mostly (as one often does also in dealing with  $\kappa$ -Minkowski) using a dual description of our q-dS spacetime that relies on an associated symmetry Hopf algebra. The relevant mathematics is not yet fully developed for the 3+1D case, and therefore we find convenient to consider primarily the cases of 2+1 and 1+1 spacetime dimensions. This is not a key limitation in light of the objectives of our analysis: rather than aiming for detailed quantitative results, we are mainly interested in exposing the presence of some interplay between Planck-scale effects and curvature, illustrating some of the typical structures to be expected for this interplay.

In Section IV we mainly argue that our results have implications that are significant even for cases in which the curvature scalar is constant, because we find that some observables, such as the distance travelled by a massless particle in a given time interval, depend on the Planck scale in measure that depends strongly on the curvature scalar. This point is at least semi-quantitatively relevant for certain observations in astrophysics, particular the ones that concern sources that are not too distant, close enough for the time variation of the curvature scalar to be negligible at least at a first level of analysis. But we expect that the interplay between curvature and Planck-scale effects should acquire even more significance in FRW-like geometries, with their associated time dependence of the curvature scalar, and we set up our case by first noticing, in Section V, that at the classical-spacetime level of analysis, one can obtain a good description of some aspects of FRW spacetimes by viewing these spacetimes as an ensemble of patches of dS spacetimes. The intuition gained in Section V then provides guidance for the analysis reported in Section VI, which is centered on the working assumption that one could get a description of a “q-FRW spacetime” by combining patches of q-dS spacetime. Section VII offers a few closing remarks on the outlook of this research area.

## II. PRELIMINARIES ON CLASSICAL DS SPACE-TIME

In preparation for our analysis it is useful to review some aspects of the classical dS spacetime, especially the description of its symmetries, the associated conserved charges, and a recipe for obtaining the path of a massless particle that relies primarily on the conserved charges.

Our notation is such that the Einstein equation, with cosmological constant  $\Lambda$ , is written as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu} \quad (3)$$

where  $R_{\mu\nu}$  is the contraction by the metric  $g_{\mu\nu}$  of the Riemann tensor,  $R$  is the Ricci scalar, and  $T_{\mu\nu}$  is the energy-momentum tensor.

In comoving coordinates the dS solution in 3+1D takes the form<sup>2</sup>

$$ds^2 = dt^2 - a_{dS}^2(t)(dx^2 + dy^2 + dz^2), \quad \text{with} \quad a_{dS}(t) = e^{Ht}. \quad (4)$$

It is a solution of (3) in empty space ( $T_{\mu\nu} = 0$ ) with  $\Lambda = 3H^2$ , but it can also be obtained in various other ways, for example as a solution of the Einstein equation without a cosmological term ( $\Lambda = 0$ ) when the energy-momentum tensor is the one for a perfect fluid with energy density  $\varrho = 3H^2/(8\pi G)$  and constant pressure  $p = -3H^2/(8\pi G)$ .

The dS solution, can be viewed as a particular FRW (Friedmann Robertson Walker) solution, in which the characteristic time-dependent conformal factor  $a(t)$  of FRW solutions takes the form  $a_{dS}(t) = e^{Ht}$ . As for all FRW solutions the spatial line element (in comoving coordinates)  $dl^2 = a^2(t)(dx^2 + dy^2 + dz^2)$  is such that the distance between two spatial points grows with time. The geodesics are orthogonal to the space like surface, and the time  $t$  is the proper time for different observers in the Universe expansion.

The dS solution is a constant-curvature spacetime, so the Riemann curvature tensor is completely determined by the Ricci scalar  $R$  through the relation  $R_{\mu\nu\alpha\beta} = \frac{R}{12}(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha})$ . Since the Ricci scalar is related to  $H$  by  $R = 12H^2$ , it is clear that the constant  $H$  suffices in order to specify the curvature of dS spacetime.

dS spacetime is conformally-flat, *i.e.* the metric is obtained by a conformal transformation from the Minkowski metric, and of course one obtains the Minkowski spacetime in the limit  $H \rightarrow 0$ . In the 3+1D case it can be described as a surface in a five-dimensional spacetime (with signature  $\{-1, 1, 1, 1, 1\}$ ) characterized by the requirement

$$-z_0^2 + z_1^2 + z_2^2 + z_3^2 + z_4^2 = H^2. \quad (5)$$

The  $SO(4, 1)$  symmetries transformations of the 3+1D dS spacetime leave unchanged the bilinear form (5), and can be viewed from the perspective of the embedding 5D spacetime as the 10 rotations which leave invariant the surface (5).

Both the symmetry generators  $G_i$  (with  $i = 0, 1, \dots, 9$ ) and the associated charges  $\Pi_i$  (conserved along a geodesic line) can be described in terms of the Killing vectors  $\xi_i^\mu$  of the metric:

$$G_i = \xi_i^\mu \partial_\mu, \quad (6)$$

$$\Pi_i = \xi_i^\mu p_\mu, \quad (7)$$

where the four-vector  $p^\mu$ , the energy-momentum measured by free-falling observers, is given for a test particle of mass  $m$  by

$$p^\mu = m \frac{dx^\mu}{d\tau} \quad (8)$$

along a geodesic with affine parameter  $\tau$ .

The Killing vectors of the metric (4) are given by

$$\begin{aligned} \xi_{P_0} &= (1, -H\vec{x}), \quad \xi_{P_1} = (0, 1, 0, 0), \quad \xi_{P_2} = (0, 0, 1, 0), \quad \xi_{P_3} = (0, 0, 0, 1), \\ \xi_{N_1} &= \left( x, \frac{1 - e^{-2Ht}}{2H} - \frac{H}{2}(x^2 - y^2 - z^2), -Hxy, -Hxz \right), \\ \xi_{N_2} &= \left( y, -Hxy, \frac{1 - e^{-2Ht}}{2H} - \frac{H}{2}(y^2 - x^2 - z^2), -Hyz \right), \\ \xi_{N_3} &= \left( z, -Hxz, -Hyz, \frac{1 - e^{-2Ht}}{2H} - \frac{H}{2}(z^2 - x^2 - y^2) \right), \\ \xi_{J_1} &= (0, 0, z, -y), \quad \xi_{J_2} = (0, -z, 0, x), \quad \xi_{J_3} = (0, y, -x, 0). \end{aligned} \quad (9)$$

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<sup>2</sup> For consistency with standard conventions used in astrophysics, here and in the following two sections we choose a time normalization such that in dS spacetime  $a_{dS} = 1$  at the present time, *i.e.* we set the present time to zero, past time to be negative and future time to be positive. We warn our readers that for Sections V-VI we shall turn to a different choice of time normalization, for consistency with the one preferred in most applications in cosmology.

We are labelling the symmetry generators in a way that refers to the Poincaré generators to which they reduce in the  $H \rightarrow 0$  limit. The generators  $G_0 \equiv P_0$ ,  $G_i \equiv P_i$ ,  $i = 1, 2, 3$  describe generalized time-like and space-like translations; the  $G_{N_i} \equiv N_i$  are dS boosts, and finally  $G_{J_i} \equiv J_i$  are rotations. The generators of course close the  $SO(4, 1)$  dS classical (Lie) algebra

$$\begin{aligned} [P_0, P_i] &= HP_i, & [P_0, N_i] &= P_i - HN_i, & [P_0, J_i] &= 0, \\ [P_i, P_j] &= 0, & [P_i, N_j] &= P_0\delta_{ij} - H\epsilon_{ijk}J_k, & [P_i, J_j] &= -\epsilon_{ijk}P_k, \\ [N_i, N_j] &= -\epsilon_{ijk}J_k, & [J_i, J_j] &= \epsilon_{ijk}J_k, & [N_i, J_j] &= -\epsilon_{ijk}N_k, \end{aligned} \quad (10)$$

with the first Casimir operator given by

$$\mathcal{C} = P_0^2 - \vec{P}^2 + H(\vec{P} \cdot \vec{N} + \vec{N} \cdot \vec{P}) - H^2 \vec{J}^2. \quad (11)$$

The conserved charges are scalars under general coordinate transformations, but one can easily verify that upon introducing formally the commutation relations

$$[p^\mu, x^\nu] = g^{\mu\nu}, \quad (12)$$

one then obtains a set of “noncommuting charges” which closes the same  $SO(4, 1)$  Lie algebra as the associated generators.

We are describing the isometries of dS spacetime in terms of a set of generators which are “natural” when using comoving coordinates. Their algebraic properties (commutators) can be viewed as properties of the “comoving-coordinates symmetry algebra”. By a general coordinates transformation an associated isomorphic realization of the symmetry algebra is found. A general coordinates transformation acts as a rotation between two different realizations of the isometry algebra.

There exists a variety of perspectives in which the motion of a particle in General Relativity can be examined. For our purposes it is useful to focus on an approach based on the representation (7) of conserved charges. We intend to focus on the motion of massless particles, which is directly connected with the causal structure of the theory. In preparation for the type of analysis described in the following section, in which we consider quantum-spacetime issues, we focus on the case of the 1+1D subalgebra of 3+1D dS algebra<sup>3</sup>.

As one can indeed infer from the analogous of (10) holding for conserved charges, the 1+1D dS algebra of the charges is characterized by the commutators

$$\begin{aligned} [\Pi_E, \Pi_p] &= H\Pi_p & [\Pi_E, \Pi_N] &= \Pi_p - H\Pi_N \\ [\Pi_p, \Pi_N] &= \Pi_E, \end{aligned} \quad (13)$$

and the first Casimir in terms of the conserved charges can be written as:

$$\mathcal{C} = \Pi_E^2 - \Pi_p^2 + H(\Pi_p\Pi_N + \Pi_N\Pi_p). \quad (14)$$

The explicit expressions of the conserved charges are

$$\Pi_E = E + Hx e^{2Ht} p, \quad (15)$$

$$\Pi_p = -e^{2Ht} p, \quad (16)$$

$$\Pi_N = x\Pi_E + \left( \frac{1 - e^{-2Ht}}{2H} + \frac{H}{2}x^2 \right) \Pi_p. \quad (17)$$

The Casimir relation (11) for the conserved charges leads to the dS mass-shell condition:

$$m^2 = E^2 - e^{2Ht} p^2. \quad (18)$$

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<sup>3</sup> The symmetry algebras for the 3+1D, 2+1D and 1+1D cases of the classical dS space-time are all contained into one another as sub-algebras.

Note that, while  $\Pi_E$  and  $\Pi_P$  are the conserved quantities, the observable one-particle energy  $E$  and momentum  $p$  are not conserved, and in particular they scale as  $E_1 a_{dS}(t_1) = E_2 a_{dS}(t_2)$ , as one can indeed infer from (18), (15) and (16) for a massless particle, consistently with the scaling induced by cosmological redshift in a dS Universe.

Let us consider now the 1+1D motion of a photon in 3+1D dS spacetime and derive the expression for the distance travelled by the photon starting at time  $-t_0$  and observed at time  $t$ . Let us notice that if  $x(-t_0) = 0$ , then

$$\Pi_N(\{x = 0, t = -t_0\}) = \mathcal{N} \equiv \frac{1 - e^{2Ht_0}}{2H} \Pi_p. \quad (19)$$

For a massless particle the Casimir equation (14) takes the form

$$\Pi_E = \pm \Pi_p \sqrt{1 - \frac{2H\mathcal{N}}{\Pi_p}}, \quad (20)$$

which can be rewritten as follows

$$\Pi_E = -\Pi_p e^{Ht_0}, \quad (21)$$

using the explicit expression (19) of  $\mathcal{N}$  (and fixing the sign ambiguity by choosing to consider a case in which  $E = p$  would hold in the  $H \rightarrow 0$  Minkowski limit).

Denoting again by  $\mathcal{N}$  the value of the conserved  $\Pi_N$  along geodesics, we can rewrite it as

$$\mathcal{N} = -\Pi_p e^{Ht_0} x + \left( \frac{1 - e^{-2Ht}}{2H} + \frac{H}{2} x^2 \right) \Pi_p, \quad (22)$$

where we have substituted  $\Pi_E$  with its expression in terms of  $\Pi_p$ , eq. (21). Solving the equation above for  $x$  we find

$$x_{dS}(t) = \frac{e^{Ht_0} \pm e^{-Ht}}{H}, \quad (23)$$

where only the minus sign is consistent with the initial condition  $x(t = -t_0) = 0$ .

### III. Q-DS AND ITS $\kappa$ -POINCARÉ/ $\kappa$ -MINKOWSKI LIMIT

In order to provide an illustrative example of the possible interplay between curvature and Planck-scale-induced quantum corrections we analyze a quantum description of dS spacetime such that its  $H \rightarrow 0$  limit provides a well-known quantum description of Minkowski spacetime, the  $\kappa$ -Minkowski noncommutative spacetime [23–28]. We find convenient to derive most results in terms of the properties of the algebra of symmetries of the quantum spacetime, rather than on the dual [29] spacetime-coordinate picture. This is the approach which turned out to be most fruitful also in the study of theories in  $\kappa$ -Minkowski [23–28]. Just as  $\kappa$ -Minkowski could be described fully as the noncommutative spacetime dual to the  $\kappa$ -Poincaré Hopf algebra [23, 24, 30], q-dS spacetime can be introduced as the spacetime dual to the q-dS Hopf algebra.

In the 3+1D q-dS case this spacetime/spacetime-symmetry picture is still only developed rather poorly [31]. We shall therefore base our intuition and obtain our results in the 2+1D and 1+1D dS cases. One indeed finds explicit formulations of the 2+1D and 1+1D q-dS Hopf algebras in the literature [30, 32], but we must stress that the relation between them is not simple, as a result of the non-embedding property of the Drinfel'd-Jimbo deformation of dS algebra. Unlike in the case of their Lie-algebra limits (mentioned in the preceding section), the 1+1D q-dS Hopf algebra cannot be obtained as a simple restriction of the 2+1D q-dS Hopf algebra, and (in spite of the preliminary nature of the results so far available on the 3+1D case) we of course expect that a

similar complication affects the relationship between the 3+1D and 2+1D cases. However, one can also see [30, 32] that the differences between the 1+1D restriction of the 2+1D q-dS Hopf algebra and the 1+1D algebra are not of a type that should lead to sharp changes in the physical picture and it is natural to expect that, once an explicit formulation for the 3+1D case will be available, the 3+1D case will also turn out to be rather similar to the other ones. One should therefore be able to obtain a rather reliable first look at q-dS theories by considering the 2+1D and even the 1+1D case.

In the next subsection we start by reviewing briefly some well-known properties of the  $\kappa$ -Minkowski spacetime that are particularly significant for our analysis. Then in Subsection III B we discuss some properties of the 2+1D q-dS Hopf algebra, focusing on the aspects that are most relevant for our analysis of the interplay between curvature and Planck-scale effects. A similar description of the 1+1D q-dS Hopf algebra is given in Subsection III C, and a few remarks on the q-dS spacetimes are offered in Subsection III D.

### A. Some key aspects of $\kappa$ -Minkowski spacetime

The  $\kappa$ -Minkowski noncommutative spacetime has coordinates that satisfy the commutation relations [23–28]

$$\begin{aligned} [x_0, x_j] &= i\lambda x_j \\ [x_j, x_k] &= 0, \end{aligned} \tag{24}$$

where the noncommutativity parameter<sup>4</sup>  $\lambda$  is often assumed to be proportional to the Planck length scale.

Even just a quick look at the commutation relations (24) already suggests that, while space-rotation symmetry remains classical, translation and boost symmetries are modified by the  $\kappa$ -Minkowski noncommutativity. These modified symmetries, as well as other properties of theories in  $\kappa$ -Minkowski [33], are very naturally described in terms of a “Weyl map” [34], a one-to-one map between elements of the space of functions of the  $\kappa$ -Minkowski noncommutative coordinates and elements of the ordinary space of functions of commuting coordinates. It is sufficient to specify such a Weyl map  $\Omega$  on the complex exponential functions and extend it to the generic function  $\phi(x)$ , whose Fourier transform is  $\tilde{\phi}(k) = \frac{1}{(2\pi)^4} \int d^4x \phi(x) e^{-ikx}$ , by linearity [33]. For instance, using a time-to-the-right ordering convention,

$$\Phi(\mathbf{x}) \equiv \Omega(\phi(x)) = \int d^4k \tilde{\phi}(k) \Omega(e^{ikx}) = \int d^4k \tilde{\phi}(k) e^{-i\vec{k} \cdot \vec{\mathbf{x}}} e^{ik_0 \mathbf{x}_0}. \tag{25}$$

(We are adopting conventions such that  $kx \equiv k_\mu x^\mu \equiv k_0 x^0 - \vec{k} \cdot \vec{\mathbf{x}}$ .)

It is relatively straightforward [33] to see that, consistently with this choice of Weyl map, the action of generators of translations,  $P_\mu$ , and space-rotations,  $M_j$ , should be described as follows

$$P_\mu \Phi(\mathbf{x}) = \Omega[-i\partial_\mu \phi(x)], \tag{26}$$

$$M_j \Phi(\mathbf{x}) = \Omega[i\epsilon_{jkl} x_k \partial_l \phi(x)]. \tag{27}$$

This means that for both translations and space-rotations one can introduce a “classical action” (classical through the Weyl map). However, while rotations are truly classical, one can easily see that (as one expects on the basis of the form of the  $\kappa$ -Minkowski commutation relations) translations are not fully classical. There is no deformation in the “action rule” (26) of translations, but a deformation necessarily appears in the “Leibnitz rule”, *i.e.* the noncommutativity scale enters in the rule for the action of translations on the product of functions of the noncommutative

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<sup>4</sup> In most of the  $\kappa$ -Minkowski literature one finds the equivalent parameter  $\kappa$ , which is  $\kappa = 1/\lambda$ , but our formulas turn out to be more compact when expressed in terms of  $\lambda$ .

coordinates. We can see this already by considering the implications of the action rule (26) for the action of translations on a product of two Fourier exponentials:

$$\begin{aligned} P_j \Omega(e^{ikx}) \Omega(e^{ipx}) &= -i \Omega(\partial_j e^{i(k+p)x}) = \\ &= -i \Omega((k+p)_j e^{i(k+p)x}) = \\ &= [P_j \Omega(e^{ikx})] [\Omega(e^{ipx})] + [e^{-\lambda P_0} \Omega(e^{ikx})] [P_j \Omega(e^{ipx})], \end{aligned} \quad (28)$$

where  $p+q \equiv (p_0 + q_0, p_1 + q_1 e^{-\lambda p_0}, p_2 + q_2 e^{-\lambda p_0}, p_3 + q_3 e^{-\lambda p_0})$  characterizes the product of exponentials in just the correct way to reflect the noncommutativity of the spacetime coordinates on which those exponentials depend on. For this type of deformations of the Leibnitz rule one speaks of the presence of a “nontrivial coproduct”. For example in the case we are now considering, the coproduct of space translations  $\Delta P_j$ , one sees from (28) that

$$\Delta P_j = P_j \otimes 1 + e^{-\lambda P_0} \otimes P_j. \quad (29)$$

Following an analogous procedure one can verify that instead the coproduct of time translations is trivial

$$\Delta P_0 = P_0 \otimes 1 + 1 \otimes P_0. \quad (30)$$

While for translations the deformation is only in the Leibnitz rule, for boosts there is even a deformation of the action rule. One finds

$$N_j \Phi(\mathbf{x}) = \Omega\left(-[ix_0 \partial_j + x_j \left(\frac{1 - e^{2i\lambda\partial_0}}{2\lambda} - \frac{\lambda}{2} \nabla^2\right) + \lambda x_l \partial_l \partial_j]\phi(x)\right). \quad (31)$$

This result can be derived in several independent ways. One possibility is just to insist on the “consistency of the Hopf algebra”: one speaks of a symmetry Hopf algebra when the commutators and coproducts of the symmetry generators close on the generators themselves. If one for example replaced (31) with the classical action of boosts then the coproduct of boosts would require [33] the introduction of operators external to the algebra.

The generators introduced in (26), (27), (31) close the well-known  $\kappa$ -Poincaré Hopf algebra [23, 24, 30]. As it is always the case for a Hopf algebra, different choices of generators for the algebra lead to formulations that are apparently rather different, the so-called different “bases” of the Hopf algebra. In the case of the  $\kappa$ -Poincaré Hopf algebra [23, 24, 30] the different bases have a simple description in terms of the ordering conventions adopted on the dual  $\kappa$ -Minkowski side. We have adopted (see (25)) the time-to-the-right convention, which is preferred by most authors [24–26, 35], and, in order to avoid potential complications which are unrelated to the point we are making, we will work throughout consistently with this choice of conventions, even at the level of the generalization to the case of q-dS algebra and spacetime.

For the generators introduced in (26), (27), (31) one obtains the following  $\kappa$ -Poincaré commutators

$$\begin{aligned} [P_\mu, P_\nu] &= 0, \\ [M_j, M_k] &= i \varepsilon_{jkl} M_l, \quad [N_j, M_k] = i \varepsilon_{jkl} N_l, \quad [N_j, N_k] = -i \varepsilon_{jkl} M_l, \\ [M_j, P_0] &= 0, \quad [M_j, P_k] = i \epsilon_{jkl} P_l, \\ [N_j, P_0] &= i P_j, \\ [N_j, P_k] &= i \left[ \left( \frac{1 - e^{-2\lambda P_0}}{2\lambda} + \frac{\lambda}{2} \vec{P}^2 \right) \delta_{jk} - \lambda P_j P_k \right], \end{aligned}$$

and coproducts

$$\begin{aligned} \Delta(P_0) &= P_0 \otimes 1 + 1 \otimes P_0, \quad \Delta(P_j) = P_j \otimes 1 + e^{-\lambda P_0} \otimes P_j, \\ \Delta(M_j) &= M_j \otimes 1 + 1 \otimes M_j, \\ \Delta(N_j) &= N_j \otimes 1 + e^{-\lambda P_0} \otimes N_j - \lambda \epsilon_{jkl} P_k \otimes M_l. \end{aligned} \quad (32)$$

Correspondingly the “mass-squared” Casimir operator,  $C_\lambda$ , takes the form

$$C_\lambda = \frac{1}{\lambda^2} [\cosh(\lambda m) - 1] = (2/\lambda)^2 \sinh^2(\lambda P_0/2) - e^{\lambda P_0} \vec{P}^2 , \quad (33)$$

where we also introduced the so called mass parameter  $m$ , which is expected to describe the rest energy.

This Casimir relation has generated significant interest in the quantum-gravity/quantum-spacetime literature. This interest originates mainly from taking the working assumption that the mass-Casimir relation between the generators  $P_0, \vec{P}$  might reflect the form of the (dispersion) relation between energy  $E$  and momentum  $p$  (expected to still be conserved charges derivable from the presence of symmetry under  $P_0, \vec{P}$ ). If this working assumption is correct then there could be some striking effects, including a dependence of speed on energy for massless particles,  $v_{(m=0)} = e^{\lambda E}$  (obtained from the dispersion relation using the familiar law  $v = dE/dp$ ). If  $\lambda$  is of the order of the Planck length, such a velocity law would fit naturally within a rather wide quantum-gravity literature which, for independent reasons, has been considering analogous laws [3, 36–38]. At energies accessible in laboratory experiments one can always safely assume  $e^{\lambda E} \simeq 1$ , but in the early stages of evolution of the Universe the typical particle energy was extremely high, and some authors have discussed [16, 19] the possibility that such laws of energy dependence of the speed of massless particles might have significant implications for our understanding of the early Universe, with significance both for inflation and possibly other features that are relevant for establishing which regions of the Universe were in causal connection at a certain era in the evolution.

Recent results [39–45] suggest that the assumption that the energy-momentum relation should exactly reproduce the Casimir relation between symmetry generators might have to be improved upon<sup>5</sup>, but also confirm that more careful analyses do not change significantly the key expectations. This provides partial encouragement for us (see later) to assume that a similar working assumption for the q-dS case can be reliably used for a first preliminary level of investigation.

Most of our more quantitative results for q-dS case will focus on the 1+1D case, so we close this subsection by noting the commutators and coproducts for the 1+1D  $\kappa$ -Poincaré Hopf algebra (written consistently with our time-to-the-right conventions):

$$\begin{aligned} [P_0, P] &= 0, & [N, P_0] &= iP, & [N, P] &= i\frac{1}{2\lambda} (1 - e^{-2\lambda P_0}) - i\frac{\lambda}{2} P^2, \\ \Delta(P_0) &= P_0 \otimes 1 + 1 \otimes P_0 , & \Delta(P) &= P \otimes 1 + e^{-\lambda P_0} \otimes P , & \Delta(N) &= N \otimes 1 + e^{-\lambda P_0} \otimes N . \end{aligned} \quad (34)$$

## B. q-dS algebra of symmetries in 2+1D

As mentioned above, we shall mainly consider q-dS algebras for 2+1D and 1+1D cases, consistently adopting throughout conventions such that these q-dS Hopf algebras contract (in the Inönü-Wigner sense) to the formulation of the  $\kappa$ -Poincaré algebra given in the conventional “time-to-the-right basis”. We shall further specify our conventions by demanding that the “classical limit” of our basis for q-dS reproduces the classical dS algebra written for comoving coordinates.

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<sup>5</sup> The studies reported in Refs. [39–45] showed that the Noether technique of derivation of conserved charges, and in particular of the energy/momentum charges associated to space/time translational invariance, are applicable also to the case of field theories with Hopf-algebra spacetime symmetries formulated in noncommutative spacetimes. There are still some challenges concerning the (“operative”) interpretation of the charges that are derived in these novel Noether analyses, but the preliminary indications that are emerging suggest that the relation between energy-momentum charges might be somewhat different from the (Casimir) relation between the translation generators with differences that however do not change the nature (order of magnitude and energy-momentum dependence) of the Planck-scale-induced correction terms. It is therefore still legitimate to perform preliminary investigation of the implications of spacetime noncommutativity assuming that the relation between charges roughly resembles the Casimir relation for the symmetry generators.

Let us start by noting down the commutators and coproducts which characterize our description of the q-dS Hopf algebra in the 2+1D case. The commutators are

$$\begin{aligned}
[J, P_0] &= 0, & [J, P_i] &= \varepsilon_{ij} P_j, & [J, N_i] &= \varepsilon_{ij} N_j, \\
[P_0, P_i] &= H P_i, & [P_0, N_i] &= P_i - H N_i, & [P_1, P_2] &= 0 \\
[P_i, N_j] &= -\delta_{ij} \left( H \frac{e^{-2w} \frac{P_0}{H}}{2w} - H \frac{\cos(2wJ)}{2w} + \frac{1}{2} \tanh(w) \left( \{P_i, N_i\} - \frac{\vec{P}^2}{H} \right) \right) + \\
&\quad - \tanh(w) \left( \frac{P_j P_i}{H} + \varepsilon_{ij} \frac{H}{2w} \sin(2wJ) - (P_j N_i + N_j P_i) \right) - \varepsilon_{ij} \frac{H}{2w} \sin(2wJ), \\
[N_1, N_2] &= -\frac{1}{2w} \sin(2wJ), 
\end{aligned} \tag{35}$$

where  $\varepsilon_{ij}$  is the Levi-Civita tensor ( $i, j \in \{1, 2\}$ ;  $\varepsilon_{12} = 1$ ) and we used notation consistent with the one introduced in the previous section for the classical limit (the  $w \rightarrow 0$  “classical” limit of our description of the q-dS Hopf algebra reproduces the description of the dS Lie algebra given in the previous section).

The coproducts are

$$\begin{aligned}
\Delta(P_0) &= 1 \otimes P_0 + P_0 \otimes 1, & \Delta(J) &= 1 \otimes J + J \otimes 1, \\
\Delta(P_i) &= e^{-w \frac{P_0}{H}} \otimes P_i + P_i \otimes \cos(wJ) - \varepsilon_{ij} P_j \otimes \sin(wJ), \\
\Delta(N_i) &= e^{-w \frac{P_0}{H}} \otimes N_i + N_i \otimes \cos(wJ) + \varepsilon_{ij} \frac{P_j}{H} \otimes \sin(wJ),
\end{aligned} \tag{36}$$

and the q-dS first Casimir is given by

$$\begin{aligned}
\mathcal{C} &= 4H^2 \cosh(w) \left[ \frac{\sinh^2 \left( \frac{wP_0}{2H} \right)}{w^2} \cos^2 \left( \frac{wJ}{2} \right) - \frac{\sin^2 \left( \frac{wJ}{2} \right)}{w^2} \cosh^2 \left( \frac{wP_0}{2H} \right) \right] - \frac{\sinh(w)}{w} e^{w \frac{P_0}{H}} \cdot \\
&\quad \cdot \left[ \cos(wJ) (\vec{P}^2 - H \{N_i, P_i\}) + 2H \sin(wJ) ((P_1 N_2 - P_2 N_1) + H \frac{1}{2w} \sin(2wJ)) \right].
\end{aligned} \tag{37}$$

This casimir relation will play a key role in our analysis. We shall analyze it mainly for what concerns the implications it suggests for conserved charges, in the spirit of the observations reported at the end of the previous subsection. Specifically we shall assume that, also in the q-dS case, the charges satisfy the same algebraic relations as the generators once the commutation relations (12) are formally introduced. Using this reasoning in reverse one can estimate the properties of the charges by looking at the ones of the generators, and taking into account the implications of introducing formally (12). We should warn our readers that in the dS case, besides the limitations of this approach already debated in the quantum-Minkowski literature (here briefly mentioned in the preceding subsection), there are additional challenges which originate in some “ordering issues”. In particular, for  $H \neq 0$  the relevant Casimir acquires a dependence on noncommuting generators. For  $H = 0$  the Casimir depends only on  $P_0$  and  $P_i$ , and they commute for  $H = 0$ . But for  $H \neq 0$  one finds that the generators  $P_0$  and  $P_i$  do not commute and in addition the Casimir also acquires a dependence on the boost generators  $N_i$  which of course does not commute with  $P_0$  and  $P_i$ . This results in an ambiguity for the implementation of the “recipe” of substitution of generators by numerical values of charges carried by a classical particle. We shall not dwell much on this “ordering issue” for  $P_0$  and  $P_i$  and present results adopting only one particular (and not necessarily compelling) choice or ordering. Some quantitative details of the formulas we produce do depend on this ordering ambiguity, but for the qualitative features we do highlight, which are the main objective of our analysis, we have verified that they are robust under changes of ordering convention.

Concerning the Inönü-Wigner contraction, which classically takes dS to Minkowski/Poincaré, it is rather significant for our analysis that one can find both contractions of q-dS to  $\kappa$ -Minkowski/ $\kappa$ -Poincaré and contractions of q-dS to classical Minkowski/Poincaré. The outcome of the Inönü-Wigner contraction procedure depends crucially on the relationship between  $H$  and the quantum-group deformation parameter  $w$ . We can easily show this feature since we have already described

the algebras in terms of appropriately “ $H$ -rescaled generators” [30, 46], and therefore the contraction will be achieved at this point by simply taking  $H \rightarrow 0$ .

For small values of  $H$  some quantum-gravity arguments (see, *e.g.*, Ref. [21] and references therein) suggest that the relation between  $H$  and  $w$  should be well approximated, for small  $H$ , by a parametrization in terms of a single parameter  $\alpha$ :

$$w \sim (HL_p)^\alpha , \quad (38)$$

where  $L_p$  is the Planck length scale ( $\simeq 10^{-33} \text{ cm}$ ) and the parameter  $\alpha$  may depend on the choice of quantum-gravity model. By inspection of the formulas given above one easily finds that, depending on the value of this parameter  $\alpha$ , the  $H \rightarrow 0$  contraction of the q-dS Hopf algebra for 2+1D spacetime leads to the following possible results:

- If  $\alpha = 1$  the contraction of the 2+1D q-dS Hopf algebra gives the 2+1D  $\kappa$ -Poincaré Hopf algebra. In particular, for  $\alpha = 1$  and small  $H$  one finds that the “q-dS mass Casimir” takes the form

$$C|_{\text{small } H} \simeq \frac{4}{L_p^2} \sinh^2 \left( \frac{L_p P_0}{2} \right) - e^{L_p P_0} \vec{P}^2 + O(H) , \quad (39)$$

which is clearly consistent with the  $\kappa$ -Poincaré mass Casimir (33). This result (39) provides an example of the case in which, at the level of infinitesimal symmetry transformations, quantum-spacetime corrections for small values of the curvature are curvature independent ( $H$ -independent). But even for these cases where  $w \simeq HL_p$  there is room for significant (see below) source of interplay between curvature and Planck-scale effects, originating from the fact that the quantum-gravity literature invites one to contemplate different case of the relationship between  $w$ ,  $H$  and  $L_p$  with the common feature of taking the shape  $w \simeq HL_p$  in (and only in) the small- $H$  limit.

- If  $1 < \alpha < 2$  the  $H \rightarrow 0$  contraction of the 2+1D q-dS Hopf algebra gives the 2+1D classical Poincaré (Lie) algebra, and for small  $H$  one finds that the “q-dS mass Casimir” takes the form

$$C|_{\text{small } H} \simeq P_0^2 - \vec{P}^2 - L_p (HL_p)^{\alpha-1} P_0 \vec{P}^2 + O(H) . \quad (40)$$

This case  $1 < \alpha < 2$  clearly is an example of very strong interplay between Planck scale and curvature, even for small curvatures (small values of  $H$ ). So much so that when  $H = 0$  there are no quantum-spacetime effects (at the symmetry-algebra level) whereas as soon as  $H \neq 0$  one finds the Planck-scale corrections. This provides very clear evidence in support of our thesis: the interplay between curvature and Planck-scale effects may be very significant, and even for small values of curvature.

- If  $\alpha \geq 2$  one stills finds (as in the case  $1 < \alpha < 2$ ) that the contraction of the 2+1D q-dS Hopf algebra gives the 2+1D classical Poincaré (Lie) algebra. But with these high values of  $\alpha$  one may say that the quantum-algebra corrections are negligible even for small but nonzero values of the curvature. In particular, for  $\alpha \geq 2$  and small  $H$  the “q-dS mass Casimir” takes the form

$$C|_{\text{small } H} \simeq P_0^2 - \vec{P}^2 . \quad (41)$$

- Finally the case  $\alpha < 1$  must be excluded since it provides an inconsistent description of the Minkowski limit: for  $\alpha < 1$  the  $H \rightarrow 0$  contraction of the 2+1D q-dS Hopf algebra is affected by inadmissible divergences. Indeed there is no known example [21] of a quantum-gravity argument favouring a relationship between  $w$ ,  $H$  and  $L_p$  characterized by  $\alpha < 1$ .

These observations show that the interplay between curvature and Planck-scale (quantum-spacetime) effects can be very significant also in the small-curvature limit. And we shall argue that this interplay can be even more significant in the large-curvature limit. Of course, in order to describe the behaviour for large values of curvature we cannot rely on (38), which is considered in the quantum-gravity literature only as a good approximation scheme for the small-curvature case. And of course the complexity of quantum-gravity theories represents a huge challenge for attempts to estimate nonperturbative features, such as the exact form of the relation between  $H$  and  $w$ . For our purposes it is however useful to adopt even a tentative *ansatz* for the exact form of the relation between  $H$  and  $w$ , since it allows us to give definite formulas that illustrate the implications of curvature for quantum-spacetime effects very explicitly, particularly by exposing differences between the small-curvature and the high-curvature regimes. With these objectives in mind we can consider the possibility

$$w = \frac{2\pi}{2 + \frac{1}{HL_p}}. \quad (42)$$

which is inspired<sup>6</sup> by results reported in the literature on 2+1D canonical quantum gravity [47–50].

The formula (42) is intriguing from our perspective since it reduces to  $w \simeq HL_p$  (*i.e.* the case considered above with  $\alpha = 1$ ) for  $H$  much smaller than  $\frac{1}{L_p}$ , but then for large values of  $H$  the quantum-algebra deformation becomes essentially constant (in the sense that  $w \simeq \pi$ ) and independent of  $L_p$ . This is therefore an example in which the case of large curvatures eliminates from our theoretical framework any dependence on the Planck scale, even though the Planck-scale effects are very significant in the small-curvature regime.

### C. q-dS algebra of symmetries and charges in 1+1D

It is easy to verify that the same type of interplay between curvature and quantum-geometry/quantum-algebra effects discussed in the previous subsection for the case of the 2+1D q-dS Hopf algebra is also found in the case of the 1+1D q-dS Hopf algebra. But we nevertheless find appropriate to report a few observations on the 1+1D q-dS Hopf algebra, since this will provide the basis for a tentative analysis of a “quantum FRW spacetime” proposed in Section VI.

Let us start by noting down some key characteristics of the 1+1D q-dS Hopf algebra [51], adopting conventions for the choice of “basis” that are consistent with the corresponding ones adopted in the previous subsections. The commutators are

$$\begin{aligned} [P_0, P] &= HP, \quad [P_0, N] = P - HN, \\ [P, N] &= \cosh(w/2) \frac{1 - e^{\frac{-2wP_0}{H}}}{2w/H} - \frac{1}{H} \sinh(w/2) e^{\frac{-wP_0}{H}} \Theta, \end{aligned} \quad (43)$$

where we introduced, for compactness, the notation

$$\Theta = \left[ e^{\frac{wP_0}{2H}} (P - HN) e^{\frac{wP_0}{2H}} (P - HN) - H^2 e^{\frac{wP_0}{2H}} N e^{\frac{wP_0}{2H}} N \right].$$

---

<sup>6</sup> We are prudently stressing that Eq. (42) for our analysis is only loosely inspired by previous results in the quantum gravity literature because of awareness of several subtleties that should be properly investigated before establishing the relevance more robustly. It is clearly encouraging for our study and for all the q-dS-based quantum-gravity research to notice that the introduction of the cosmological constant in 2+1D canonical quantum gravity allows the resulting gauge symmetry of the theory to be described in terms of Hopf algebras/quantum groups [47, 50]. So the quantum groups symmetries are, in an appropriate sense, not an *a priori* choice of hypothesis for these models, but rather something that is constructively derived. However, for what concerns specifically the relation between  $w$  and  $H$  codified in Eq. (42) we should stress that this is found [49] specifically in the study of 2+1D gravity with negative cosmological constant rewritten as a Chern-Simons theory.

For the coproducts one finds

$$\Delta(P_0) = 1 \otimes P_0 + P_0 \otimes 1, \quad \Delta(P) = e^{\frac{-wP_0}{H}} \otimes P + P \otimes 1, \quad \Delta(N) = e^{\frac{-wP_0}{H}} \otimes N + N \otimes 1, \quad (44)$$

and the mass Casimir is

$$\mathcal{C} = H^2 \frac{\cosh(w/2)}{w^2/4} \sinh^2 \left( \frac{wP_0}{2H} \right) - \frac{\sinh(w/2)}{w/2} \Theta. \quad (45)$$

In Section VI we examine some properties of the propagation of massless particles in a quantum FRW spacetime on the basis of some corresponding properties of massless particles in 1+1D q-dS spacetime. We are therefore primarily interested in analyzing (43),(45) for the case of a massless particle. In particular, from the form of the Casimir we infer that for massless particles

$$0 = H^2 \frac{\cosh(w/2)}{w^2/4} \sinh^2 \left( \frac{wP_0}{2H} \right) - \frac{\sinh(w/2)}{w/2} \Theta, \quad (46)$$

and therefore

$$\sinh(w/2) \Theta = H^2 \frac{\cosh(w/2)}{w/2} \sinh^2 \left( \frac{wP_0}{2H} \right). \quad (47)$$

This last equation can be used to obtain a simplified form of the last commutator in (43):

$$[P, N] = \frac{H}{w} \cosh(w/2) \left( e^{\frac{-wP_0}{H}} - e^{\frac{-2wP_0}{H}} \right). \quad (48)$$

As discussed earlier, these properties at the algebra level can be used to motivate some corresponding proposals for the conserved charges, also using formally the commutation relations  $[p^\mu, x^\nu] = g^{\mu\nu}$ . This leads us to the following representation of the charges<sup>7</sup>:

$$\Pi_E = E + H e^{2Ht} x p, \quad \Pi_p = \partial_x = -e^{2Ht} p, \quad (49)$$

$$\begin{aligned} \Pi_N = & \frac{H}{2w} \frac{\cosh(w/2)}{\sinh w} e^{-\frac{w\Pi_E}{H}} \frac{1}{\Pi_P} e^{-\frac{w\Pi_E}{H}} (e^{-2w\Pi_P x} - 1) + \\ & - \frac{H}{2w} \frac{1}{\tanh(w/2)} e^{-\frac{w\Pi_E}{2H}} \frac{1}{\Pi_P} e^{-\frac{w\Pi_E}{2H}} (e^{-w\Pi_P x} - 1) + \frac{1 - e^{-2Ht}}{2H} \Pi_P. \end{aligned} \quad (50)$$

Notice that the representations of  $\Pi_E$ ,  $\Pi_p$  are unchanged with respect to the corresponding classical-spacetime case. The quantum-algebra deformation only affects the representation of  $\Pi_N$ . Of course, also  $\Pi_N$  reduces to its classical-spacetime limit upon setting  $w \rightarrow 0$

$$\lim_{w \rightarrow 0} \Pi_N = \frac{1}{2} H x^2 \Pi_p + x \Pi_E + \frac{1 - e^{-2Ht}}{2H} \Pi_P. \quad (51)$$

As discussed in Section II, the  $\Pi_N$  charge plays a key role in the derivation of the path of a massless particle, and therefore one should expect that the quantum-algebra deformation of the representation of the  $\Pi_N$  charge affects significantly the description of the path of a massless particle. As in Section II we observe that

$$\mathcal{N} \equiv \Pi_N(\{x = 0, t = -t_0\}) = \frac{1 - e^{2Ht_0}}{2H} \Pi_P. \quad (52)$$

---

<sup>7</sup> In order to verify that the charges described in (49) and (50) (once formally endowed with the noncommutativity implied by the commutation relations  $[p^\mu, x^\nu] = g^{\mu\nu}$ ) close the q-dS Hopf algebra it is sufficient to make use of the Sophus-Lie expansion and of the observation  $[A, \frac{1}{B}] = -\frac{1}{B}[A, B]\frac{1}{B}$ , which is valid for any pair of noncommuting operators  $A$  and  $B$ .

And enforcing this constraint (52) into the Casimir relation one obtains

$$0 = H^2 \frac{\cosh(w/2)}{w^2/4} \sinh^2 \left( \frac{w\Pi_E}{2H} \right) - \frac{\sinh(w/2)}{w/2} \Theta_{\mathcal{N}}, \quad (53)$$

where  $\Theta_{\mathcal{N}} = \left[ e^{\frac{w\Pi_E}{H}} (\Pi_P - H\mathcal{N})^2 - H^2 e^{\frac{w\Pi_E}{H}} \mathcal{N}^2 \right]$ .

Solving the above equation with respect to the variable  $Y \equiv e^{-\frac{w\Pi_E}{H}}$  one finds two solutions:

$$Y_{\pm} = 1 \pm \sqrt{\frac{2w}{H^2} \tanh(w/2) [(\Pi_P - H\mathcal{N})^2 - H^2 \mathcal{N}^2]}, \quad (54)$$

which, using (52), can be rewritten in the form

$$Y_{\pm} = 1 \pm \frac{e^{Ht_0} \Pi_P}{H} \sqrt{2w \tanh(w/2)}. \quad (55)$$

We shall only take into account the solution  $Y_+$ , ignoring  $Y_-$ , since we are looking for  $\Pi_P = -\Pi_E$  in the  $w \rightarrow 0, H \rightarrow 0$  limit (see Sec. II). In the following we denote simply with  $Y$  the solution  $Y_+$ .

Since  $\Pi_N$  is conserved during the particle motion, we require it to be always equal to  $\mathcal{N}$ . From the charge definition (50) one then obtains

$$\mathcal{N} = \frac{H}{2w} \frac{1}{2 \sinh \frac{w}{2}} \frac{Y^2}{\Pi_P} (e^{-2w\Pi_P x} - 1) - \frac{H}{2w} \frac{1}{\tanh(w/2)} \frac{Y}{\Pi_P} (e^{-w\Pi_P x} - 1) + \frac{1 - e^{-2Ht}}{2H} \Pi_P, \quad (56)$$

and solving with respect to  $e^{-w\Pi_P x}$  one finds

$$\begin{aligned} e^{-w\Pi_P x} &= \cosh \frac{w}{2} Y^{-1} - \sqrt{\left(1 - \cosh \left(\frac{w}{2}\right) Y^{-1}\right)^2 + 2 \frac{w}{H} \sinh \left(\frac{w}{2}\right) \Pi_P Y^{-2} \left(-\frac{1 - e^{-2Ht}}{H} \Pi_P + 2\mathcal{N}\right)} \\ &= \cosh \frac{w}{2} Y^{-1} - \sqrt{\left(1 - \cosh \left(\frac{w}{2}\right) Y^{-1}\right)^2 + 2w \sinh \left(\frac{w}{2}\right) \Pi_P^2 Y^{-2} \frac{e^{-2Ht} - e^{2Ht_0}}{H^2}}, \end{aligned} \quad (57)$$

where we also used (52) to eliminate  $\mathcal{N}$ , and we removed a sign ambiguity by enforcing consistency with the initial condition  $x(t = -t_0) = 0$ .

So the deformed comoving distance travelled by a q-dS mass-less particle, that starts moving at time  $t = -t_0$  is:

$$x_{q-dS}(t) = -\frac{1}{w\Pi_P} \ln \left[ \cosh \frac{w}{2} Y^{-1} - \sqrt{\left(1 - \cosh \left(\frac{w}{2}\right) Y^{-1}\right)^2 + 2w \sinh \left(\frac{w}{2}\right) \Pi_P^2 Y^{-2} \frac{e^{-2Ht} - e^{2Ht_0}}{H^2}} \right]. \quad (58)$$

This formula has the correct  $w \rightarrow 0$  limit, since in this limit it reduces to the comoving distance travelled by a massless particle in dS spacetime (see eq. (23))

$$\lim_{w \rightarrow 0} x_{q-dS} = \frac{e^{Ht_0} - e^{-Ht}}{H} + O(w). \quad (59)$$

Since  $Y$  denotes the  $Y_+$  of Eq. (55) (and therefore  $Y$  depends only on  $\Pi_P, w, H$ ), our result (58) gives the dependence of  $x_{q-dS}(t)$  on  $\Pi_P, w, H$  and  $t$ . Of course, if preferred, one can also use Eq. (55) to trade the dependence on  $\Pi_P$  for a dependence on  $\Pi_E$ , obtaining

$$x_{q-dS}(t) = \frac{\sqrt{2w \tanh \left(\frac{w}{2}\right)}}{w H e^{-Ht_0} \left(1 - e^{\frac{-w\Pi_E}{H}}\right)} \ln [Z] \quad (60)$$

with  $Z = \cosh \left(\frac{w}{2}\right) e^{\frac{w\Pi_E}{H}} - \sqrt{\left(1 - \cosh \left(\frac{w}{2}\right) e^{\frac{w\Pi_E}{H}}\right)^2 + \cosh \left(\frac{w}{2}\right) e^{-2Ht_0} \left(1 - e^{\frac{-w\Pi_E}{H}}\right)^2 (e^{-2Ht} - e^{2Ht_0})}$ .

#### D. Aside on quantum dS space-time

We structure our analysis in such a way that we can rely exclusively on the structure of the q-dS Hopf algebra of symmetries, without any explicit use of the noncommutativity of the q-dS spacetime. This, as mentioned, is consistent with an approach that has proven fruitful in the analysis of other spacetimes that are dual to a Hopf algebra, such as  $\kappa$ -Minkowski. Still some readers may find more intuitive a characterization of the spacetime which is not only implicitly given in terms of a duality. In closing this section we therefore provide an explicit description of the q-dS noncommutative spacetime, relying on results previously obtained in the literature [52].

The simplest strategy for obtaining an explicit description of the properties of the q-dS spacetime coordinates uses a procedure that performs a semiclassical quantization of the Poisson-Lie brackets [52], based on the familiar Weyl substitution [53, 54] of the Poisson brackets between commutative coordinates by commutators between non-commutative coordinates. While one can have quantum groups that do not coincide with the Weyl quantization of its underlying Poisson-Lie brackets, this procedure has proven fruitful in several previous applications (see, *e.g.*, Refs. [55, 56] and references therein).

In Ref. [52] one finds an explicit description of the Poisson-Lie brackets for the 2+1D dS algebra<sup>8</sup>:

$$\{x_0, x_1\} = -w \frac{\tan Hx_1}{H^2 \cos^2 Hx_2}, \quad \{x_0, x_2\} = -w \frac{\tanh Hx_2}{H^2}, \quad \{x_1, x_2\} = 0.$$

From these one obtains the commutation rules for the coordinates of the 2+1D q-dS spacetime

$$\begin{aligned} [\hat{x}_0, \hat{x}_1] &= -w \frac{\tan H\hat{x}_1}{H^2 \cos^2 H\hat{x}_2} + o(w^2) = -\frac{w}{H}\hat{x}_1 - \frac{1}{3}wH\hat{x}_1^3 - wH\hat{x}_1\hat{x}_2^2 + o(w^2, H^2), \\ [\hat{x}_0, \hat{x}_2] &= -w \frac{\tan H\hat{x}_2}{H^2} + o(w^2) = -\frac{w}{H}\hat{x}_2 - \frac{1}{3}wH\hat{x}_2^3 + o(w^2, H^2), \\ [\hat{x}_1, \hat{x}_2] &= 0 + o(w^2). \end{aligned} \tag{61}$$

As mentioned the q-dS spacetime reduces to  $\kappa$ -Minkowski when an appropriate  $H \rightarrow 0$  limit is taken. Indeed if one assumes in (61) that for small  $H$  the quantization parameter  $w$  is proportional to  $H$ , *i.e.*  $w \simeq \lambda H$  for some  $\lambda$ , then the  $H \rightarrow 0$  limit of the commutation relations (61) reproduces the commutation relations of the  $\kappa$ -Minkowski spacetime coordinates.

One may also introduce a description of the q-dS spacetime in terms of non-commutative ambient (Weierstraß) coordinates  $(\hat{s}_3, \hat{s}_\mu)$ , which reads [52]

$$\begin{aligned} [\hat{s}_0, \hat{s}_i] &= -\frac{w}{H}\hat{s}_3\hat{s}_i + o(w^2), & [\hat{s}_1, \hat{s}_2] &= 0 + o(w^2), \\ [\hat{s}_3, \hat{s}_0] &= -wH\hat{s}^2 + o(w^2), & [\hat{s}_3, \hat{s}_i] &= -wH\hat{s}_0\hat{s}_i + o(w^2). \end{aligned} \tag{62}$$

In this formulation the symmetry under exchange of  $\hat{s}_1$  and  $\hat{s}_2$  is manifest, and, since  $\hat{s}_3 \rightarrow 1$  when  $H \rightarrow 0$ , the first two relations in (62) are directly connected to the corresponding properties of the  $\kappa$ -Minkowski coordinates.

#### E. Some possible applications of the q-dS algebra

The research effort we report in this manuscript was aimed at establishing as robustly as possible the significance of the interplay between curvature and Planck-scale effects, thereby correcting a

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<sup>8</sup> Notice that the asymmetric form of the brackets with respect to exchanges of coordinates is not an intrinsic feature of the theoretical framework but rather a result [52] of the particular choice of the local coordinates  $x_\mu$ .

commonly-adopted assumption in quantum-gravity-phenomenology research. In a certain sense the q-dS formalism is viewed within our analysis as a toy model that is well suited for exposing fully our concerns that it is not legitimate to assume absence of interplay between curvature and Planck-scale effects. In closing this Section on the q-dS formalization we find appropriate to stress that we feel that this formalization may well deserve more interest than the one from the toy-model perspective, although we shall not impose this intuition on our readers elsewhere in the manuscript.

One of the reasons for our choice to focus nearly exclusively on the significance of the interplay between curvature and Planck-scale effects is that this aspect has emerged from our investigations as a fully robust feature, qualitatively independent of the choices of perspective and ordering conventions we adopted. Up to relatively uninteresting quantitative details the same significance of interplay between curvature and Planck-scale effects is easily found even adopting choices of ordering convention that are different from the one on which we focused for simplicity. And similarly one finds exactly the same level of interplay between curvature and Planck-scale effects upon changing the basis for the q-dS Hopf algebra, going for example from the one we here preferred (because of its relevance for the much-studied “time-to-the-right basis” of the  $\kappa$ -Poincaré algebra) to one obtained even by nonlinear redefinition of the generators.

This robustness of the significance of the interplay between curvature and Planck-scale should have profound implications for the directions to be taken in parts of the quantum-gravity-phenomenology literature, but it is of course not at all surprising within the framework we adopted. As we stressed already in the opening remarks of this manuscript, the striking feature of the q-dS framework is that the key novel structures depend on a single parameter  $w$  which by construction is dimensionless. So the only opportunities for the Planck scale to appear in the description of the structure of our quantum spacetime necessarily involve expressing this dimensionless parameter in terms of the Planck scale and of the only other dimensionful scale present in the framework, which is indeed the curvature scalar.

Of course it is also interesting to examine the q-dS framework looking for features that are of interest even beyond the issue of establishing the presence of a strong interplay between curvature and Planck-scale effects. Our perception is that these specific features might be more sensitive to possible “changes of Hopf-algebra basis” and possible alternative ways to handle the ordering issues discussed above. But they are nonetheless interesting and we want to comment on at least a couple of them.

Probably the most significant of these features concerns the possibility of describing a “minimum-wavelength principle” in a framework that allows for curvature. The idea of a “minimum-wavelength principle” is justifiably popular in the quantum-gravity literature and in fact several flat-spacetime formalizations have been proposed, but to our knowledge the framework we developed here is the first example of a possible description of a “minimum-wavelength principle” in presence of curvature. One way to see this is based on our equation (55) which (since we worked with  $Y \equiv e^{-\frac{w\Pi_E}{H}}$  and  $\Pi_p = -|\Pi_p|$ ) establishes that

$$e^{-\frac{w\Pi_E}{H}} = 1 - \frac{e^{Ht_0}|\Pi_P|}{H} \sqrt{2w \tanh(w/2)} . \quad (63)$$

Let us assume for definiteness that  $w = HL_p$  and let us first notice that this equation produces a “minimum-wavelength principle” in the  $H \rightarrow 0$  limit, which is indeed the “minimum-wavelength principle” that motivated a significant portion of the interest in the  $\kappa$ -Minkowski/ $\kappa$ -Poincaré framework:

$$e^{-L_p\Pi_E} = 1 - L_p|\Pi_P| . \quad (64)$$

This indeed reflects the known mechanism for exposing a minimum wavelength (maximum  $|\Pi_P|$ ) in the  $\kappa$ -Minkowski/ $\kappa$ -Poincaré framework: the maximum allowed value for  $|\Pi_P|$  is  $|\Pi_P| = 1/L_p$ , as  $|\Pi_P|$  approaches the value  $1/L_p$  the flat-spacetime energy  $\Pi_E$  diverges, and for hypothetical values of  $|\Pi_P|$  greater than  $1/L_p$  there is no real-energy solution. For values of  $H$  different from 0 one should probably not attach much intrinsic significance to the details of (63), which are going to depend on the mentioned issues of choice of Hopf-algebra basis and choice of ordering prescription, but still (63) allows us to raise a significant point: a minimum-wavelength principle,

when implemented in a quantum geometry with curvature, must take into account the effects of redshift (illustrated in (63) through the presence of the factor  $e^{Ht_0}$ ).

Another feature which can be meaningfully discussed in relation to the broader quantum-gravity/quantum-spacetime literature is the one of “ultraviolet-infrared mixing”. It is expected that the short distance structure introduced for spacetime quantization could (and perhaps should [57]) also affect the long-wavelength (infrared) regime [58, 59]. Some trace of infrared manifestations of the short-distance quantum-geometry structure of our q-dS setup is present, although in rather implicit form, in the result for the deformed comoving distance travelled by a massless particle we obtained in Eq. (58) (and these infrared features will be exposed more explicitly in reanalyses of (58) which we discuss later on in this manuscript). We shall not dwell here on the quantitative details of these infrared features, since they too should depend on the mentioned issues of choice of Hopf-algebra basis and choice of ordering prescription, but we still feel that the possibility to investigate ultraviolet-infrared mixing in a quantum geometry with curvature is very exciting. In particular, flat-spacetime ultraviolet-infrared mixing, while attracting much interest at the level of its technical description, is still confronted by severe challenges of interpretation, because in a flat quantum spacetime the only characteristic scale is the Planck scale, an ultraviolet scale that clearly cannot on its own govern the onset of infrared features. In our q-dS framework instead the curvature scalar and the Planck scale inevitably cooperate, opening the way to more realistic descriptions of ultraviolet-infrared mixing. The (energy) scale  $H$  clearly could govern infrared features, and perhaps more excitingly the framework also naturally allows for contemplating a role for the (squared-energy) scale  $H/L_p$ , which is some sort of hybrid between the ultraviolet and the infrared structure of the spacetime geometry, thereby potentially offering a particularly natural candidate for the scale characteristic of the infrared side of the mechanism of ultraviolet-infrared mixing.

#### IV. AN APPLICATION IN ASTROPHYSICS

The formalization developed in the previous section gives a definite picture for the interplay of curvature and Planck-scale effects in dS-like (constant Hubble parameter) quantum geometries, and clearly should also provide a meaningful first approximation applicable to contexts in astrophysics that involve sources at relatively small distances (small redshift,  $z < 1$ ), since then the analysis only involves rather small time variations of the Hubble parameter. And relevant for our thesis is the fact that a much-studied possible implication of Planck-scale Hopf-algebra symmetries, for which often results on  $\kappa$ -Minkowski theory provide at least part of the motivation, is the one of a dependence of the speed of massless particles on energy. There is sizeable interest in particular in studies [3–12] of this hypothesis of energy-dependent speed for photons that exploit the nearly ideal “laboratory” provided by observations of gamma-ray bursts. The sensitivity to “in-vacuo dispersion” (fundamental energy dependence of the speed of massless particles) of gamam-ray-burst studies is rather significant, in spite of the limitations imposed by the fact that the source, the “gamma-ray burster”, has intrinsic time variability [10–12]. Previous related phenomenology work assumed no interplay between curvature and Planck-scale effects, so that from a Hopf-algebra perspective it would amount to assuming that the energy dependence found in the  $\kappa$ -Minkowski case should be added by hand to the analysis of particle propagation in classical FRW spacetime. We shall instead rely on the results reported in the previous two sections to provide a preliminary characterization of particle propagation in a quantum (noncommutative) curved spacetime.

For these purposes we shall of course rely on our analysis of  $x_{q-dS}(t)$ , the comoving distance travelled by a q-dS photon in a time interval  $t$ . Since  $L_p$  is small and, in the applications in astrophysics that can be here of interest,  $H$  is also small, we can assume that our dependence of the parameter  $w$  on  $HL_p$  should be analyzed for small values of  $HL_p$ . And for all the scenarios we considered for the relation  $w = f(HL_p)$  one finds that  $w$  is small when  $HL_p$  is small. We can

therefore rely on an approximation of our result for  $x_{q-dS}(t)$  valid<sup>9</sup> for small  $w$ :

$$x_{q-dS}^{(app)}(t) = \frac{e^{Ht_0} - e^{-Ht}}{H} - w \frac{H^2 - 4e^{-2Ht}\Pi_P^2 + 4e^{2Ht_0}\Pi_P^2 - e^{H(t+t_0)}H^2}{8H^2\Pi_P}. \quad (65)$$

In the ideal case of two photons emitted simultaneously by a very compact source, a photon with momentum  $p$  and a photon with momentum  $\ll p$ , we shall assume that the soft photon is detected at time  $t = 0$  while the photon of momentum  $p$  is detected at some time  $t = \delta_{qds}$ . The two photons would have covered the same comoving distance, which, since for the soft photon we can neglect quantum-spacetime effects [10, 11], we can denote by  $x_{dS}(0)$ . This allows us to derive the delay time  $\delta_{qds}$  from the following equation:

$$x_{dS}(0) = x_{q-dS}^{(app)}(\delta_{qds})|_{\Pi_p = -pe^{2H\delta_{qds}}}. \quad (66)$$

Working in leading order in  $\delta_{qds}$  one then obtains

$$\delta_{qds} = \frac{(-H^2 + 4p^2 - 4e^{2Ht_0}p^2 + e^{Ht_0}H^2)w}{8H^2p} \simeq \frac{p(1 - e^{2Ht_0})w}{2H^2}, \quad (67)$$

where on the right-hand side we took into account of the smallness of the values of  $H$  that are relevant for gamma-ray-burst studies ( $H \sim 10^{-33}$  eV).

The relevance for our thesis of this result (67) originates from the explicit dependence of  $\delta_{qds}$  on  $H$  and perhaps even more significantly from the implicit dependence on  $H$  contained in  $w$ . The Planck length  $L_p$  only appears in  $\delta_{qds}$  through  $w$ , and therefore the interplay between curvature and Planck-scale effects is very significant. The part of this interplay codified in the fact that necessarily  $w$  must be written in terms of  $HL_p$  is evidently a robust feature of our framework, qualitatively independent of the mentioned issues concerning the choice of Hopf-algebra basis and the choice of ordering prescription. The residual dependence on  $H$  (such as the factor  $e^{2Ht_0}$ ) is instead a more fragile aspect of our analysis, but still indicative of the type of qualitative features that one in general should expect.

Also notice that (67) reflects the requirement  $\alpha \geq 1$  that emerged in the analysis we reported in the previous section, for cases with  $w \simeq (HL_p)^\alpha$ . In fact, for  $\alpha < 1$  the  $H \rightarrow 0$  limit of (67) is pathological (whereas no pathology arises for  $\alpha \geq 1$ ).

## V. DS SLICING OF A FRW UNIVERSE

Our results on a quantum-dS spacetime dual to a quantum/Hopf algebra of symmetries are of potential relevance also for the astrophysics of distant sources ( $z \gtrsim 1$ ) and for cosmology, but only in a rather indirect way. We shall argue that taking as starting point our dS-like quantum spacetime represents an advantage of perspective with respect to the most common strategy adopted so far in the study of Planck-scale effects in astrophysics and cosmology, which relies on taking as starting point results on the quantization of Minkowski-like spacetimes. The most evident advantage is of course due to the fact that at least we can make use of some intuition on the interplay between curvature and Planck-scale effects, which was indeed the primary motivation for our study.

Before actually applying our strategy to quantum-spacetime contexts, we find useful to devote this Section to a sort of test of this strategy in a classical-spacetime context, where of course it is

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<sup>9</sup> Notice that Eq. (65) formally has a singularity for  $\Pi_P \rightarrow 0$ . This is an example of the infrared features mentioned in Subsection III E, and, as we argued, one should not necessarily interpret it as an artifact of our approximations. It is known that in noncommutative spacetimes (like our q-dS noncommutative spacetime) such infrared issues may arise, and could be a meaningful manifestation of the novel uncertainty principle for the localization of spacetime points that a quantum spacetime predicts. It is nonetheless reassuring that our results are however well behaved whenever  $\Pi_P \geq H$  (and we are clearly not interested in particles with  $\Pi_P < H$ , since  $H_0 \sim 10^{-33}$  eV).

easier for us (and for our readers) to confidently assess its efficacy. Specifically, in this Section we introduce, at the classical level, an approximation of a FRW solution in which the time axis is divided in small  $\Delta t$  intervals and in each  $\Delta t$  interval the FRW Universe is approximated by a corresponding “slice” of the dS space-time, and we compare our findings to the corresponding results one obtains by using analogously “Minkowski slices” (which is the classical-spacetime version of the strategy used in most studies of Planck-scale effects in astrophysics and cosmology). It is obvious *a priori* that both “slicing strategies” must converge to a faithful description of the FRW results in the limit of very detailed slicing (locally, in an infinitesimal neighborhood of a point in FRW both approximations are exact of course). But it is interesting to find confirmation that even at the classical level the dS slicing, through its ability of codifying information on curvature, is a better tool of approximation.

It is sufficient for our purposes to contemplate the case of the FRW solution whose line-element is

$$ds^2 = dt^2 - a^2(t) d\vec{x}^2. \quad (68)$$

In order to provide explicit formulas it is useful for us to fix the conformal factor  $a(t)$ , and we take as illustrative example the case  $a(t) = a_{rad-3D}(t) \equiv (t/t_0)^{2/3}$ , where  $t_0$  is a normalization time<sup>10</sup>. While in this section this choice of  $a(t)$  is as good as any other possible illustrative example, but we shall again adopt this choice of  $a(t)$  in the next section, which is where we report the aspects of our analysis that are potentially relevant for cosmology. The choice  $a(t) = (t/t_0)^{2/3}$  is adopted there by the desire to contemplate the first radiation-dominated instants of evolution of the Universe, while taking into account that most of our insight on the theory side, and particularly the *ansatz* reported in Eq. (42), originated in 2+1D theories: indeed one finds that  $a(t) = (t/t_0)^{2/3}$  in a radiation-dominated 2+1D FRW Universe.

Let us then split the time interval  $\{0, t_F\}$ , in  $n$  small intervals  $\Delta t = t_F/n$ , with  $n \in \mathbb{N}$ . In each  $k$ -th interval  $\Delta t$  we consider a corresponding “slice” of dS space-time, with time variable  $t' \in [0, \Delta t]$ , spatial coordinate  $\vec{x}'$  and with line element given by

$$ds_k^2 = dt'^2 - A_k^2 e^{2H_k t'} d\vec{x}'^2. \quad (69)$$

Here  $H_k$  is related to the square root of the “effective cosmological constant” relative to the  $k$ -th interval, and the additional parameter, the scaling constant  $A_k$ , must also appropriately match some FRW requirements. Specifically, in order to match the FRW evolution given by  $a(t)$ , we must fix  $A_k$  and  $H_k$ , in each  $k$ -th interval of duration  $\Delta t$ , through the following requirements:

$$a(t = m\Delta t + t') = \sum_{k=1}^n A_k e^{H_k t'} \theta[(k+1)\Delta t - t] \theta[t - k\Delta t], \quad (70)$$

where the product of the two heavyside functions selects the  $m$ -th term in the summation.

For  $a(t) = a_{rad-3D}(t) \equiv (t/t_0)^{2/3}$  this leads to:

$$A_k = \left( \frac{k\Delta t}{t_0} \right)^{2/3}, \quad (71)$$

$$H_k = \frac{2}{3\Delta t} \ln \left( \frac{k+1}{k} \right), \quad (72)$$

where we have imposed  $A_k = a_{rad-3D}(k\Delta t)$  and  $a_{rad-3D}(k\Delta t + t')|_{t'=\Delta t} = a_{rad-3D}((k+1)\Delta t)$ .

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<sup>10</sup> In this and the next section we set  $t = 0$  at the Big Bang time.

With the approximated  $a(t)$  one can calculate the physical distance travelled by a FRW photon in a time  $t_F = n_F \Delta t$ , which is given by<sup>11</sup>

$$\begin{aligned} l_{FRW}^{(dS)}(t_F) &= a(t_F) \int_0^{t_F} \frac{dt}{a(t)} = a(t_F) \sum_{k=1}^{n_F} \int_0^{\Delta t} \frac{dt'}{a(k\Delta t + t')} = \\ &= (t_F)^{\frac{2}{3}} \sum_{k=1}^{n_F} \frac{1}{(k\Delta t)^{\frac{2}{3}}} \frac{1 - e^{-H_k \Delta t}}{H_k} = (t_F)^{\frac{2}{3}} \sum_{k=1}^{n_F} \frac{1}{(k\Delta t)^{\frac{2}{3}}} [x_{dS}(\Delta t)]_k , \end{aligned} \quad (73)$$

where for  $[x_{dS}(\Delta t)]_k$  we use Eq. (23) (with  $t_0 = 0$  and  $H = H_k$  given by Eq. (72)).

Within our approximation, Eq. (73), the distance travelled by the FRW photon is expressed in terms of the formula that gives the comoving distance travelled by a dS photon. In Planck units, if  $t_F = 1$  the exact distance travelled by the FRW photon is  $l_{FRW}(1) = 3$ , which also encodes the “horizon paradox” (the distance travelled by the FRW photon starting from  $t = 0$  up to a time  $t_F$  is always smaller than what would be needed in order to achieve agreement with the observed isotropy of the CMBR). We of course reproduce faithfully this feature in our “dS slicing of FRW” in the limit of vanishingly small slices:

$$\lim_{\Delta t \rightarrow 0} l_{FRW}^{(dS)}(1) = 3 . \quad (74)$$

In order to test the accuracy of our approximation (73) in the evaluation of the distance travelled by a FRW photon, it is useful to compare  $l_{FRW}^{(dS)}(t_F)/a(t_F)$  and  $l_{FRW}/a(t_F)$ . For this purpose we observe that

$$\begin{aligned} \Delta l^{(dS)} &\equiv \frac{l_{FRW}^{(dS)}(t_F)}{a(t_F)} - \frac{l_{FRW}}{a(t_F)} = t_0^{\frac{2}{3}} \sum_{k=1}^{n_F} \frac{1}{(k\Delta t)^{\frac{2}{3}}} \frac{1 - e^{-H_k \Delta t}}{H_k} - t_0^{\frac{2}{3}} \sum_{k=1}^{n_F} \int_{\Delta t k}^{\Delta t(k+1)} \frac{dt'}{t'^{\frac{2}{3}}} \\ &= 3 t_0^{\frac{2}{3}} \sum_{k=1}^{n_F} \Delta t^{\frac{1}{3}} \left[ \frac{1}{2 \log(1 + \frac{1}{k})} \left( \frac{(\frac{1}{k} + 1)^{\frac{2}{3}} - 1}{(k+1)^{\frac{2}{3}}} \right) - (k+1)^{\frac{1}{3}} + k^{\frac{1}{3}} \right] , \end{aligned} \quad (75)$$

from which one easily infers the exact agreement achieved for  $\Delta t \rightarrow 0$ .

It is interesting for us to examine how our approximation, for finite  $\Delta t$ , compares with an analogous approximation based on “Minkowski slices”. In a Minkowski slicing the line element in each  $k$ -th interval has the form:

$$ds_k^2 = dt'^2 - B_k^2 dx'^2 , \quad (76)$$

with  $B_k = (k\Delta t/t_0)^{2/3}$ . And the associated natural approximation of  $l_{FRW}(t_F)$  is

$$l_{FRW}^{(M)}(t_F) = t_F^{\frac{2}{3}} \sum_{k=1}^{n_F} \frac{\Delta t^{\frac{1}{3}}}{k^{\frac{2}{3}}} . \quad (77)$$

The accuracy of this Minkowski-slicing approximation can be inferred from examining

$$\Delta l^{(M)} \equiv \frac{l_{FRW}^{(M)}(t_F)}{a(t_F)} - \frac{l_{FRW}}{a(t_F)} = t_0^{\frac{2}{3}} \sum_{k=1}^{n_F} \Delta t^{\frac{1}{3}} \left[ \frac{1}{k^{\frac{2}{3}}} - 3(k+1)^{\frac{1}{3}} + 3k^{\frac{1}{3}} \right] \quad (78)$$

And it is noteworthy that each  $k$ -th element of this summation is bigger than the corresponding  $k$ -th element of the dS-slicing case (75). Since all the terms in these summations are positive, one then concludes that the total difference between the physical distance calculated exactly in FRW and the one calculated through Minkowski slicing is always bigger than corresponding difference between the exact distance and the one calculated through dS slicing.

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<sup>11</sup> Notice that the lowest value of  $k$  is  $k = 1$  (rather than  $k = 0$  as one could perhaps naively imagine). This, as shown later in this section, ensures that our “slicing procedure” converges (in the vanishingly-thin-slice limit) to a faithful description of the physical distance travelled by a FRW photon. If one naively added the term with  $k = 0$  the result would be pathologically divergent, since  $\lim_{k \rightarrow 0} (k\Delta t)^{-2/3} \left[ 3\Delta t / \ln(1 + \frac{1}{k})^2 \right] \left( 1 - 1/\sqrt{1 + \frac{1}{k}} \right) = \infty$ .

## VI. Q-DS SLICING OF A Q-FRW UNIVERSE

In this section we contemplate a possible application of our scheme of analysis in a regime where curvature does take large values and therefore the interplay between Planck-scale and curvature can be particularly significant. This is the context of studies of the propagation of massless particles in the early Universe, focused primarily on its implications for causality. We shall proceed following a strategy which is inspired by the observations we reported in the previous section. Since the Hopf-algebra/noncommutative-spacetime literature does not offer candidates<sup>12</sup> for a “q-FRW Universe” (a FRW-like quantum spacetime) we shall rely on the assumption that propagation of massless particles in such a spacetime admits approximation in terms of “q-dS slicing”, so that we can once again rely on our result for the q-dS comoving distance.

Clearly for the application to the early Universe that we propose in this section it is more difficult to gauge the size of the inaccuracies and fragilities introduced by our approximations and choices of ordering prescriptions. We still expect our analysis to display the qualitatively correct nature of the interplay between curvature and Planck scale in the early Universe (according to the general framework inspired by  $\kappa$ -Minkowski and q-dS), but quantitatively the approximations we produce may well eventually turn out to be rather poor. Still we feel that the lessons learned through our analysis are valuable, especially in light of the fact that the first pioneeristic  $\kappa$ -Minkowski-inspired studies of the early Universe produced so far (see, *e.g.* Refs. [16–19] and references therein) completely neglect the possibility of interplay between curvature and Planck-scale effects.

Our case for the significance of the interplay between curvature and Planck-scale effects in the early Universe is based on a description of the motion of a photon in a quantum FRW Universe obtained as a summation of terms given by our proposal for the q-dS comoving distance. And our “q-dS slicing” assumes a description of  $a(t)$  which, as already done and motivated in the previous section, is the one appropriate for a radiation-dominated era in 2+1D cosmology,  $a(t) = a_{rad}(t) = (t/t_0)^{2/3}$ . As long as  $t$  is small but still  $t > 1$  in Planck units,  $a_{rad}(t)$  is a good approximation of the evolution of such a 2+1D Universe. For  $t \lesssim 1$  one clearly expects new physics to come into the picture. But for our exploratory purposes we choose to simply adopt  $a_{rad}(t)$  even for values of  $t$  all the way down to  $t = 0$ , where according to the classical setting the primordial Big-Bang point, a singularity in the Riemann tensor, is found. This setup will allow us in particular to produce observations that are directly relevant for a research programme [16–19] which explores the possibility of introducing Planck-scale-modified laws of propagation as a way to replace inflation in solving some of the cosmological paradoxes. We shall not ourselves dwell on whether or not these proposals are promising, but we rather intend to provide new tools that could play a role in future investigations of the effectiveness of these proposals.

In light of these preliminary considerations our starting point clearly must be equation (73), rewritten for the case of q-dS slicing (rather than the original dS slicing) through the substitution

$$[x_{dS}(\Delta t)]_k \longmapsto [x_{q-dS}(\Delta t)]_k. \quad (79)$$

For  $[x_{q-dS}(\Delta t)]_k$  we can rely on Eq. (60), with  $H = H_k$  and  $t_0 = 0$ :

$$[x_{q-dS}(t)]_k = \frac{\sqrt{2w \tanh(\frac{w}{2})}}{wH_k(1 - e^{\frac{-w\Pi_E}{H_k}})} \ln [Z_k], \quad (80)$$

where  $Z_k$  can be written as

$$Z_k = \cosh\left(\frac{w}{2}\right) e^{\frac{w\Pi_E}{H_k}} - \sqrt{\left(1 - \cosh\left(\frac{w}{2}\right) e^{\frac{w\Pi_E}{H_k}}\right)^2 + \cosh\left(\frac{w}{2}\right) \left(1 - e^{\frac{-w\Pi_E}{H_k}}\right)^2 (e^{-2H_k t} - 1)}.$$

<sup>12</sup> Within the Loop-Quantum-Gravity approach there have been recent proposals [60–62] of quantum-geometry descriptions of the early Universe. The type of issues that we are here concerned with has not yet been studied within this Loop-Quantum-Gravity approach, but insightful results have been obtained for example in investigations of the possibility that such quantum geometries may be suitable for a description of the early Universe that is free from a  $t = 0$  singularity.

Of course, one also needs a procedure for “slice matching”: our  $[x_{q-dS}(dt)]_k$  depends on  $\Pi_E$ , which is a conserved quantity in our q-dS framework, but clearly would not be a good conserved charge in a quantum FRW Universe. This issue of the determination of  $\Pi_E$  in each  $k$ -th interval is clearly related to the scaling of energy in a quantum FRW Universe. This is a key point where, because of the unavailability of a formulation of quantum FRW spacetime, we can only proceed by adopting a plausible *ansatz*, thereby loosing control on the quantitative accuracy of our estimates. However, as stressed already in other points of this manuscript, our primary objectives are not of detailed quantitative nature. Specifically, in this section on implications for the early Universe we simply want to provide some support for our main point that the interplay between curvature and Planck-scale effects can be particularly significant at high curvature. And we base our thesis not on the details of what we find for one particular choice of  $w(HL_p)$  (the dependence of the deformation parameter  $w$  on  $HL_p$ ), but rather on the comparison between the results obtained for two different but related choices of  $w(HL_p)$ . In light of this our (qualitative) findings are relatively insensitive to changes of the choice of *ansatz* used to fix  $\Pi_E$  in each  $k$ -th interval.

The *ansatz* we adopt is inspired by the scaling of the energy  $E_{in} = E_F[a(t_F)/a(t_{in})]$  in a classical FRW universe. We further estimate the scale factor by considering the FRW radiation dominated era  $a_{rad}(t) = (t/t_0)^{2/3}$ . This in turn leads us to assuming that at the initial time of each  $k$ -th interval, within our “q-dS slicing”,  $[\Pi_E]_k$  should be tentatively described as follows:

$$[\Pi_E]_k \propto \frac{E_F t_F^{2/3}}{(k \Delta t)^{\frac{2}{3}}}, \quad (81)$$

which has already been specialized to our illustrative example of time dependence of the scale factor  $a(t) = (t/t_0)^{2/3}$ .

With this *ansatz* we specified the only unknown of our “slicing procedure” for the description of a quantum FRW spacetime in terms of our findings for the q-dS spacetime. In principle this can be used for preliminary calculations of physical distances over which a photon propagates in a certain chosen time interval. But it appears that some striking indications of the curvature dependence of Planck-scale effects, our main objective here, can be found even without embarking in a numerical analysis. We can do this by a comparison of two of the cases for the relationship between  $w$ ,  $H$  and  $L_p$  which we motivated in Section III, specifically the case<sup>13</sup>  $w = 2\pi HL_p$  and the case  $w = \frac{2\pi}{2+1/HL_p}$ . This is interesting because for small values of  $H$  one has  $\frac{2\pi}{2+1/HL_p} \simeq 2\pi HL_p$  but for large values of  $H$  one has that  $\frac{2\pi}{2+1/HL_p} \neq 2\pi HL_p$ , and therefore the formula  $w = \frac{2\pi}{2+1/HL_p}$  (which, as mentioned in Section III, is one of the few all-order formulas of this type that finds some support at least in one quantum-gravity approach) is itself a probe of the possible relevance of the curvature dependence of Planck-scale quantum-spacetime effects.

To expose the sought curvature dependence of the Planck-scale effects it suffices to examine the  $\Delta t \rightarrow 0$  limit of the generic  $k$ -th term in our description for the physical distance in q-FRW, which is given by

$$\lim_{\Delta t \rightarrow 0} \frac{t_F^{\frac{2}{3}} [x_{q-dS}(\Delta t)]_k}{(k \Delta t)^{\frac{2}{3}}}. \quad (82)$$

For the case  $w = 2\pi HL_p$ , also taking into account the behaviour of  $[\Pi_E]_k$  given in (81), one finds that for small  $\Delta t$  (“fine slicing”)

$$\frac{t_F^{\frac{2}{3}} [x_{q-dS}(\Delta t)]_k}{(k \Delta t)^{\frac{2}{3}}} \simeq \frac{t_F^{\frac{4}{3}}}{k^{\frac{4}{3}}} 2\sqrt{\pi L_p} E_F \frac{(\Delta t)^{1/6}}{(2/3 \log(1 + \frac{1}{k}))^{3/2}}. \quad (83)$$

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<sup>13</sup> Note that in Section III, because of the objectives of that part of our analysis, we were satisfied to consider generically the possibility  $w \propto HL_p$ , while here our desire to establish a more precise connection with the alternative choice  $w = \frac{2\pi}{2+1/HL_p}$  leads us to contemplate specifically the case  $w = 2\pi HL_p$ . In this way we arrange a comparison that is particularly insightful since  $\frac{2\pi}{2+1/HL_p} \approx 2\pi HL_p$  for small  $H$ , while  $\frac{2\pi}{2+1/HL_p} \ll 2\pi HL_p$  for  $H \gg 1/L_p$ .

Looking at the same quantity for the case  $w = \frac{2\pi}{2+1/HL_p}$  one finds instead, again for small  $\Delta t$ ,

$$\frac{t_F^{\frac{2}{3}} [x_{q-dS}(\Delta t)]_k}{(k\Delta t)^{\frac{2}{3}}} \simeq \frac{t_F^{\frac{2}{3}}}{k^{\frac{2}{3}}} \frac{\sqrt{\tanh(\pi/2)} (\cosh(\pi/2))^2}{\cosh(\pi/2) - 1} \frac{3}{\sqrt{2\pi} \log(1 + \frac{1}{k})} (\Delta t)^{1/3} \quad (84)$$

The fact that curvature affects the quantum properties of our q-dS spacetime, in ways that mainly originate from the dependence of  $w$  on  $H$ , is here reflected in the fact that two different but related *ansätze*,  $w = 2\pi HL_p$  and  $w = \frac{2\pi}{2+1/HL_p}$  produce different descriptions of the physical distances over which a photon propagates in a given time interval. The main differences are encoded in the different dependence on the “slice label”  $k$ , since  $k$  labels primarily  $H_k$  and the two scenarios we are comparing have different dependence of  $w$  on  $H$  (*i.e.*  $w_k$  on  $H_k$ ). Also notice that associated to this different  $k$  dependence one also finds a different dependence on  $\Delta t$ , the “thickness” of the slice in the time direction. This is ultimately where the most profound causality-relevant implications should be found, since then by taking the  $\Delta t \rightarrow 0$  limit of sums of terms of this sort one would compute the overall distance travelled by the photon. For some choices of the dependence of  $w$  on  $H$  and  $L_p$  one should expect to even find that the physical distance over which a photon propagates in a given finite interval can diverge, as essentially assumed (without however considering a possible role for curvature) in Refs. [16–19]. We shall not dwell on these possibilities here since we perceive in the limited scopes of our analysis (particularly for what concerns choice of ordering prescription and the *ansatz* (81)) an invitation to be prudent at the level of quantitative predictions, but we do feel that our findings are robust for what concerns the significance of the interplay between curvature and Planck-scale effects in early-Universe cosmology, here qualitatively exposed by comparing Eqs. (83) and (84).

## VII. OUTLOOK

The interplay between curvature and Planck-scale effects, whose significance was here robustly established, could well gradually acquire a key role in quantum-gravity-inspired studies in astrophysics and cosmology. Within a fully specified model of course one would have a definite prediction for this interplay, which could be very valuable from a phenomenology perspective, since it would provide a more distinctive characterization of the physical implications of the model. This could be exploited for example in cases where the candidate quantum-gravity effect of interest is subject to “competition” by other new-physics proposals (*see, e.g.*, Ref. [63]). While these could indeed be valuable opportunities produced by the interplay between curvature and Planck scale, it is likely that at least for some quantum-gravity/quantum-spacetime models (depending on the specific form of interplay that a given model will predict) this interplay will also introduce some challenges. For example, the fate of causality in quantum spacetime is of course a major concern, and the preliminary analysis we reported in Section VI suggests that the implications for causality of some of these quantum-gravity-inspired models might be amplified in contexts involving large curvature, such as the description of the early Universe.

The model dependence of these challenges and opportunities should be explored within the q-dS framework and other possible formalizations of scenarios with curved quantum spacetimes. For the q-dS framework we here exposed some key aspects of this model dependence, which mainly concern possible ambiguities originating from changes of ordering prescription and nonlinear redefinitions of the basis of Hopf-algebra generators. Of course, it would be desireable to show that the physical predictions of the q-dS framework do not depend on these apparently arbitrary choices, and some results obtained for flat quantum spacetimes (and their Hopf algebras of symmetries) provide encouragement [39, 43] for this hope. But also in this respect the presence of curvature may introduce some challenges, which should be addressed in dedicated studies.

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